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CERTAIN GENERATING RELATIONS ASSOCIATED WITH THE FAMILY OF BESSEL FUNCTIONS AND MULTIPLE HYPERGEOMETRIC FUNCTIONS

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Abstract: In this paper, we obtain Laurent and Maclaurin types of hypergeometric generating relations for certain mixed special functions related to the Bessel functions using series rearrangement technique. Further some generating functions for Appell's functions, Lauricella's functions and multiple hypergeometric functions of Kampé de Fériet, are also given.

Keywords and Phrases: Appell functions, Bessel functions, Generating relations, Kampé de Fériet's multiple hypergeometric functions, Lauricella functions.

2020 Mathematics Subject Classification: 33C65, 33C10, 05A15.

1. Introduction and Preliminaries

The analytical and numerical study of generalized Bessel functions has revealed their interesting properties, which in some sense can be regarded as an extension of the properties of Bessel functions to a two-dimensional domain. In this connection, the relevance of generalized Bessel function and their multi-variable extension in mathematical physics have been emphasized, since they provide analytical solutions to partial differential equations such as the multi-dimensional diffusion equation, the Schrödinger and Klein-Gordon equations. The algebraic structure underlying generalized Bessel functions can be recognized in full analogy with Bessel functions, and may provide a unifying view to the theory of both Bessel and generalized Bessel functions. Hence the interest for the generalized Bessel functions is justified.

1-variable Bessel functions:

1-variable Bessel functions $J_n(x)$ [13, p.113, Th.(39)] are defined by the following generating function:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)\ t^n, \qquad t \neq 0. \tag{1.1}$$

1-variable Modified Bessel functions:

1-variable Bessel functions $I_n(x)$ [13, p.122, Q.No.(17)] are defined by the following generating function:

$$\exp\left[\frac{x}{2}\left(t+\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} I_n(x)\ t^n, \quad t \neq 0. \tag{1.2}$$

2-variable Bessel functions:

2-variable Bessel functions $J_n(x,y)$ [6, p.332, Eq.(2.7)] see also [10, p.271, Eq.(1.1)] are defined by the following generating function:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]\exp\left[\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x,y)\ t^n. \tag{1.3}$$

2-variable Hermite-Bessel functions:

Bessel functions ${}_HJ_n(x,y)$ [10, p.277, Eq.(4.4)] are defined by the following generating function:

$$\exp\left[x\left(t-\frac{1}{t}\right)\right]\exp\left[\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)-\frac{1}{4}\left(t-\frac{1}{t}\right)^2\right] = \sum_{n=-\infty}^{+\infty} {}_HJ_n(x,y)\ t^n. \tag{1.4}$$

3-variable Hermite-Bessel functions associated with the Bell-type polynomials:

3-variable Hermite-Bessel functions ${}_{H^{(3,2)}}J_n(x,w,y)$ [7, p.405, Eq.(36)] are defined by the following generating function:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]\exp\left[\frac{w}{4}\left(t-\frac{1}{t}\right)^2\right]\exp\left[\frac{y}{8}\left(t-\frac{1}{t}\right)^3\right] = \sum_{n=-\infty}^{+\infty} {}_{H^{(3,2)}}J_n(x,w,y)\ t^n. \tag{1.5}$$

The explicit form of ${}_{H^{(3,2)}}J_n(x,w,y)$ are given as:

$$ {}_{H^{(3,2)}}J_n(x,w,y) = \sum_{r=0}^{\infty} \frac{(-1)^r \ H_{n+2r}^{(3,2)}(x,w,y)}{r! \ (n+r)! \ 2^{n+2r}}, $$

where $H_n^{(3,2)}(x,w,y)$ are defined by

$$ H_n^{(3,2)}(x,w,y) = n! \ \sum_{r=0}^{[\frac{n}{3}]} \frac{y^r \ H_{n-3r}^{(2)}(x,w)}{r! \ (n-3r)!}, $$

$$ H_n^{(2)}(x,y) = \sum_{r=0}^{[\frac{n}{2}]} \frac{n! \ y^r x^{n-2r}}{r! \ (n-2r)!} = x^n \ {}_2F_0 \left[\begin{array}{ccc} \frac{-n}{2}, & \frac{-n+1}{2}; & \\ & & y\left(\frac{-2}{x}\right)^2 \\ & - \quad ; & \end{array} \right] = g_n^2(x,y), $$

which is known as Gould-Hopper polynomials [16, p.76, Eq.(6)].

$$ H_n(x) = \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r n! \ (2x)^{n-2r}}{r! \ (n-2r)!} = (2x)^n \ {}_2F_0 \left[\begin{array}{ccc} \frac{-n}{2}, & \frac{-n+1}{2}; & \\ & & -\frac{1}{x^2} \\ & - \quad ; & \end{array} \right] = g_n^2(2x,-1). $$

Pochhammer symbol:
In our investigations, we shall use the following standard notations:
$\mathbb{N} := \ \{1,2,3,\cdots\}\,; \mathbb{N}_0 := \mathbb{N}\bigcup\{0\}\,; \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}\,.$
The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively. The Pochhammer symbol $(\alpha)_p (\alpha, p \in \mathbb{C})$ [13, p.22, Eq.(1), p.32, Q.N.(8) and Q.N.(9)], see also [16, p.23, Eq.(22) and Eq.(23)] is defined by

$$ (\alpha)_p := \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} = \begin{cases} 1 & ;(p=0; \alpha \in \mathbb{C}\backslash\{0\}), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & ;(p=n \in \mathbb{N}; \alpha \in \mathbb{C}), \\ \frac{(-1)^n k!}{(k-n)!} & ;(\alpha=-k; p=n; n,k \in \mathbb{N}_0; 0 \le n \le k), \\ 0 & ;(\alpha=-k; p=n; n,k \in \mathbb{N}_0; n > k), \\ \frac{(-1)^n}{(1-\alpha)_n} & ;(p=-n; n \in \mathbb{N}; \alpha \in \mathbb{C}\backslash\mathbb{Z}). \end{cases} $$

It being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Multiple hypergeometric functions of Kampé de Fériet:
For $(a_A) = a_1, a_2.., a_A$ and $\left(b^{(i)}_{B^{(i)}}\right) = b^{(i)}_1, b^{(i)}_2, b^{(i)}_3, ..., \ b^{(i)}_{B^{(i)}}$, we consider the multiple hypergeometric series [14]

$$F^{A:\ B^{(1)};\cdots;B^{(n)}}_{C:\ D^{(1)};\cdots;D^{(n)}}\left[\begin{array}{lr} (a_A):\left(b^{(1)}_{B^{(1)}}\right);\cdots;\left(b^{(n)}_{B^{(n)}}\right); & \\ & x_1,...,x_n \\ (c_C):\left(d^{(1)}_{D^{(1)}}\right);\cdots;\left(d^{(n)}_{D^{(n)}}\right); & \end{array}\right]$$

$$= \sum_{m_1,m_2,...,m_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m_1+m_2+\cdots+m_n} \prod_{j=1}^{B^{(1)}}\left(b^{(1)}_j\right)_{m_1} \cdots \prod_{j=1}^{B^{(n)}}\left(b^{(n)}_j\right)_{m_n}}{\prod_{j=1}^{C}(c_j)_{m_1+m_2+\cdots+m_n} \prod_{j=1}^{D^{(1)}}\left(d^{(1)}_j\right)_{m_1} \cdots \prod_{j=1}^{D^{(n)}}\left(d^{(n)}_j\right)_{m_n}} \prod_{i=1}^{n}\frac{x_i^{m_i}}{m_i!}, \quad (1.6)$$

which unifies and extends the four Lauricella series $F^{(n)}_A, F^{(n)}_B, F^{(n)}_C$ and $F^{(n)}_D$ in n variables. Infact, as already observed in literature ([9], [12] and see also [15, pp.37-38]), the multiple hypergeometric series (1.6), is a special case of the generalized Lauricella series in several variables, which was introduced by Srivastava and Daoust in 1969.
Suppose

$$\Delta_k \equiv 1 + C + D^{(k)} - A - B^{(k)} \qquad (k = 1, ..., n).$$

For the convergence [18, p.1127, Eq.(4.3)-Eq.(4.5)] of the multiple hypergeometric series (1.6), we have

$$(i)\Delta_k > 0; \quad |x_1| < \infty, \quad |x_2| < \infty \ , ..., \ |x_n| < \infty, \qquad (1.7)$$

$$(ii)\Delta_k = 0; \ \ A > C \ \ and \ \ |x_1|^{\frac{1}{(A-C)}} + \cdots + |x_n|^{\frac{1}{(A-C)}} < 1, \qquad (1.8)$$

$$(iii)\Delta_k = 0; \ \ A \le C \ \ and \ \ \max\{|x_1|, ..., |x_n|\} < 1. \qquad (1.9)$$

In order to get the clear idea about the absolutely and conditionally convergence of multiple series (1.6), see [8, pp.113-114, Th.(4)-Th.(6)].
Decomposition of bilateral series:

$$\sum_{n=-\infty}^{+\infty}\Phi(n) = \sum_{n=-\infty}^{+\infty}\Phi(2n) + \sum_{n=-\infty}^{+\infty}\Phi(2n+1), \qquad (1.10)$$

provided that both sides are absolutely convergent.

Combinatorial identity:

$$\frac{1}{p!}(\lambda)_p = \binom{\lambda+p-1}{p}. \tag{1.11}$$

Motivated by the work done by Agarwal et al. [1, 2], Chen et al. [4], Korsch et al. [11] and Srivastava et al. [17]. The present article is organized as follows: In section 2, we obtain eight generating relations following with proof by using , one, two and three variable Bessel functions and multiple hypergeometric functions of Kampé de Fériet. In section 3 we discuss some special cases.

2. Laurent and Maclaurin Types Hypergeometric Generating Relations

When $t \neq 0$, then following results hold true.

(i)

$$\cosh\left[\frac{x}{2}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\right]\exp\left[\frac{y}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_{2n}(x,y)\; t^n. \tag{2.1}$$

(ii)

$$\sinh\left[\frac{x}{2}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\right]\exp\left[\frac{y}{2}\left(t-\frac{1}{t}\right)\right] = \sqrt{t}\sum_{n=-\infty}^{+\infty} J_{2n+1}(x,y)\; t^n. \tag{2.2}$$

(iii)

$$\cosh\left[x\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\right]\exp\left[\frac{y}{2}\left(t-\frac{1}{t}\right)-\frac{1}{4}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2\right] = \sum_{n=-\infty}^{+\infty} {}_HJ_{2n}(x,y)\; t^n. \tag{2.3}$$

(iv)

$$\sinh\left[x\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\right]\exp\left[\frac{y}{2}\left(t-\frac{1}{t}\right)-\frac{1}{4}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2\right] = \sqrt{t}\sum_{n=-\infty}^{+\infty} {}_HJ_{2n+1}(x,y)t^n. \tag{2.4}$$

(v)

$$\cosh\left[\frac{x}{2}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)+\frac{y}{8}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3\right]\exp\left[\frac{w}{4}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2\right]$$
$$= \sum_{n=-\infty}^{+\infty} {}_{H^{(3,2)}}J_{2n}(x,w,y)\; t^n. \tag{2.5}$$

(vi)

$$\sinh\left[\frac{x}{2}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)+\frac{y}{8}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3\right]\exp\left[\frac{w}{4}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2\right]$$
$$=\sqrt{t}\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n+1}(x,w,y)\;t^n. \tag{2.6}$$

(vii)

$$(1-t)^{-\lambda}F^{1+A:D^{(1)};\cdots;D^{(n)}}_{B:E^{(1)};\cdots;E^{(n)}}\left[\begin{array}{c} \lambda,(a_A):\left(d^{(1)}_{D^{(1)}}\right);\left(d^{(2)}_{D^{(2)}}\right);\cdots;\left(d^{(n)}_{D^{(n)}}\right); \\ (b_B):\left(e^{(1)}_{E^{(1)}}\right);\left(e^{(2)}_{E^{(2)}}\right);\cdots;\left(e^{(n)}_{E^{(n)}}\right); \end{array}\ \frac{x_1}{1-t},\frac{x_2}{1-t},\cdots,\frac{x_n}{1-t}\right]$$
$$=\sum_{p=0}^{\infty}\binom{\lambda+p-1}{p}\times$$
$$\times F^{1+A:D^{(1)};\cdots;D^{(n)}}_{B:E^{(1)};\cdots;E^{(n)}}\left[\begin{array}{c} \lambda+p,(a_A):\left(d^{(1)}_{D^{(1)}}\right);\left(d^{(2)}_{D^{(2)}}\right);\cdots;\left(d^{(n)}_{D^{(n)}}\right); \\ (b_B):\left(e^{(1)}_{E^{(1)}}\right);\left(e^{(2)}_{E^{(2)}}\right);\cdots;\left(e^{(n)}_{E^{(n)}}\right); \end{array}\ x_1,x_2,\cdots,x_n\right]t^p. \tag{2.7}$$

(viii)

$$(1-t)^{-\lambda}F^{A:1+D^{(1)};\cdots;D^{(n)}}_{B:E^{(1)};\cdots;E^{(n)}}\left[\begin{array}{c} (a_A):\lambda,\left(d^{(1)}_{D^{(1)}}\right);\left(d^{(2)}_{D^{(2)}}\right);\cdots;\left(d^{(n)}_{D^{(n)}}\right); \\ (b_B):\left(e^{(1)}_{E^{(1)}}\right);\left(e^{(2)}_{E^{(2)}}\right);\cdots;\left(e^{(n)}_{E^{(n)}}\right); \end{array}\ \frac{x_1}{1-t},\frac{x_2}{1-t},\cdots,\frac{x_n}{1-t}\right]$$
$$=\sum_{p=0}^{\infty}\binom{\lambda+p-1}{p}\times$$
$$\times F^{A:1+D^{(1)};\cdots;D^{(n)}}_{B:E^{(1)};\cdots;E^{(n)}}\left[\begin{array}{c} (a_A):\lambda+p,\left(d^{(1)}_{D^{(1)}}\right);\left(d^{(2)}_{D^{(2)}}\right);\cdots;\left(d^{(n)}_{D^{(n)}}\right); \\ (b_B):\left(e^{(1)}_{E^{(1)}}\right);\left(e^{(2)}_{E^{(2)}}\right);\cdots;\left(e^{(n)}_{E^{(n)}}\right); \end{array}\ x_1,\frac{x_2}{1-t},\cdots,\frac{x_n}{1-t}\right]t^p, \tag{2.8}$$

provided that both sides of each equation are convergent.

Proof of relation (2.5)

From equations (1.5) and (1.10), we have

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{8}\left(t-\frac{1}{t}\right)^3\right]\exp\left[\frac{w}{4}\left(t-\frac{1}{t}\right)^2\right]$$
$$=\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n}(x,w,y)t^{2n}+\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n+1}(x,w,y)t^{2n+1}. \tag{2.9}$$

Put $t=i\sqrt{T},\ t^2=-T$ in equation (2.9), we get

$$\exp\left[\frac{x}{2}\left(i\sqrt{T}+\frac{i}{\sqrt{T}}\right)+\frac{y}{8}\left(i\sqrt{T}+\frac{i}{\sqrt{T}}\right)^3\right]\exp\left[\frac{w}{4}\left(i\sqrt{T}+\frac{i}{\sqrt{T}}\right)^2\right]$$
$$=\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n}(x,w,y)(-T)^n+i\sqrt{T}\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n+1}(x,w,y)(-T)^n,$$

$$\exp\left[i\left\{\frac{x}{2}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)-\frac{y}{8}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)^3\right\}\right]\exp\left[-\frac{w}{4}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)^2\right]$$
$$=\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n}(x,w,y)(-T)^n+i\sqrt{T}\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n+1}(x,w,y)(-T)^n,$$

$$\left\{\cos\left[\frac{x}{2}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)-\frac{y}{8}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)^3\right]+\right.$$
$$\left.+i\sin\left[\frac{x}{2}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)-\frac{y}{8}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)^3\right]\right\}\exp\left[-\frac{w}{4}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)^2\right]$$

$$=\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n}(x,w,y)(-T)^n+i\sqrt{T}\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n+1}(x,w,y)(-T)^n. \tag{2.10}$$

Now equating real and imaginary parts of equation (2.10), we get real part as

$$\cos\left[\frac{x}{2}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)-\frac{y}{8}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)^3\right]\exp\left[-\frac{w}{4}\left(\sqrt{T}+\frac{1}{\sqrt{T}}\right)^2\right]$$
$$=\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n}(x,w,y)(-T)^n. \tag{2.11}$$

Put $T=-t$ or $-T=t$, $\sqrt{T}=i\sqrt{t}$ in equation (2.11), we get

$$\cos\left[\frac{x}{2}\left(i\sqrt{t}-\frac{i}{\sqrt{t}}\right)-\frac{y}{8}\left(i\sqrt{t}-\frac{i}{\sqrt{t}}\right)^3\right]\exp\left[-\frac{w}{4}\left(i\sqrt{t}-\frac{i}{\sqrt{t}}\right)^2\right]$$
$$=\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n}(x,w,y)t^n,$$

$$\cos i\left[\frac{x}{2}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)+\frac{y}{8}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3\right]\exp\left[\frac{w}{4}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2\right]$$
$$=\sum_{n=-\infty}^{+\infty}{}_{H^{(3,2)}}J_{2n}(x,w,y)t^n.$$

After further simplification, we get the relation (2.5).

Proof of relation (2.7)

Let

$$\Upsilon=(1-t)^{-\lambda}\times$$
$$\times F^{1+A:D^{(1)};\cdots;D^{(n)}}_{B:E^{(1)};\cdots;E^{(n)}}\left[\begin{array}{l}\lambda,(a_A):\left(d^{(1)}_{D^{(1)}}\right);\left(d^{(2)}_{D^{(2)}}\right);\cdots;\left(d^{(n)}_{D^{(n)}}\right);\\ \\ (b_B):\left(e^{(1)}_{E^{(1)}}\right);\left(e^{(2)}_{E^{(2)}}\right);\cdots;\left(e^{(n)}_{E^{(n)}}\right);\end{array}\quad \frac{x_1}{1-t},\frac{x_2}{1-t},\cdots,\frac{x_n}{1-t}\right]$$

$$=(1-t)^{-\lambda}\times$$
$$\times\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\frac{(\lambda)_{m_1+m_2+\cdots+m_n}(a_1)_{m_1+m_2+\cdots+m_n}(a_2)_{m_1+m_2+\cdots+m_n}\cdots\ (a_A)_{m_1+m_2+\cdots+m_n}}{(b_1)_{m_1+m_2+\cdots+m_n}(b_2)_{m_1+m_2+\cdots+m_n}\cdots\ (b_B)_{m_1+m_2+\cdots+m_n}}\times$$
$$\times\frac{\left(d^{(1)}_1\right)_{m_1}\left(d^{(1)}_2\right)_{m_1}\cdots\left(d^{(1)}_{D^{(1)}}\right)_{m_1}\left(d^{(2)}_1\right)_{m_2}\left(d^{(2)}_2\right)_{m_2}\cdots\left(d^{(2)}_{D^{(2)}}\right)_{m_2}\cdots}{\left(e^{(1)}_1\right)_{m_1}\left(e^{(1)}_2\right)_{m_1}\cdots\left(e^{(1)}_{E^{(1)}}\right)_{m_1}\left(e^{(2)}_1\right)_{m_2}\left(e^{(2)}_2\right)_{m_2}\cdots\left(e^{(2)}_{E^{(2)}}\right)_{m_2}\cdots}\times$$
$$\times\frac{\cdots\left(d^{(n)}_1\right)_{m_n}\left(d^{(n)}_2\right)_{m_n}\cdots\left(d^{(n)}_{D^{(n)}}\right)_{m_n}(x_1)^{m_1}(x_2)^{m_2}\cdots(x_n)^{m_n}}{\cdots\left(e^{(n)}_1\right)_{m_n}\left(e^{(n)}_2\right)_{m_n}\cdots\left(e^{(n)}_{E^{(n)}}\right)_{m_n}(1-t)^{m_1}m_1!(1-t)^{m_2}m_2!\cdots(1-t)^{m_n}\,m_n!}$$
$$=\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\frac{(\lambda)_{m_1+m_2+\cdots+m_n}(a_1)_{m_1+m_2+\cdots+m_n}(a_2)_{m_1+m_2+\cdots+m_n}\cdots(a_A)_{m_1+m_2+\cdots+m_n}}{(b_1)_{m_1+m_2+\cdots+m_n}(b_2)_{m_1+m_2+\cdots+m_n}\cdots(b_B)_{m_1+m_2+\cdots+m_n}}\times$$

$$\times \frac{\left(d_1^{(1)}\right)_{m_1} \left(d_2^{(1)}\right)_{m_1} \cdots \left(d_{D^{(1)}}^{(1)}\right)_{m_1} \left(d_1^{(2)}\right)_{m_2} \left(d_2^{(2)}\right)_{m_2} \cdots \left(d_{D^{(2)}}^{(2)}\right)_{m_2} \cdots}{\left(e_1^{(1)}\right)_{m_1} \left(e_2^{(1)}\right)_{m_1} \cdots \left(e_{E^{(1)}}^{(1)}\right)_{m_1} \left(e_1^{(2)}\right)_{m_2} \left(e_2^{(2)}\right)_{m_2} \cdots \left(e_{E^{(2)}}^{(2)}\right)_{m_2} \cdots} \times$$

$$\times \frac{\cdots \left(d_1^{(n)}\right)_{m_n} \left(d_2^{(n)}\right)_{m_n} \cdots \left(d_{D^{(n)}}^{(n)}\right)_{m_n} (x_1)^{m_1}(x_2)^{m_2} \cdots (x_n)^{m_n}}{\cdots \left(e_1^{(n)}\right)_{m_n} \left(e_2^{(n)}\right)_{m_n} \cdots \left(e_{E^{(n)}}^{(n)}\right)_{m_n} m_1!m_2! \cdots m_n!} \times$$

$$\times\ {}_1F_0 \left[\begin{array}{ll} \lambda + m_1 + m_2 + \cdots + m_n; & \\ & t \\ - & ; \end{array} \right]$$

$$= \sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} \frac{(\lambda)_{m_1+m_2+\cdots+m_n} (a_1)_{m_1+m_2+\cdots+m_n} (a_2)_{m_1+m_2+\cdots+m_n} \cdots (a_A)_{m_1+m_2+\cdots+m_n}}{(b_1)_{m_1+m_2+\cdots+m_n} (b_2)_{m_1+m_2+\cdots+m_n} \cdots (b_B)_{m_1+m_2+\cdots+m_n}} \times$$

$$\times \frac{\left(d_1^{(1)}\right)_{m_1} \left(d_2^{(1)}\right)_{m_1} \cdots \left(d_{D^{(1)}}^{(1)}\right)_{m_1} \left(d_1^{(2)}\right)_{m_2} \left(d_2^{(2)}\right)_{m_2} \cdots \left(d_{D^{(2)}}^{(2)}\right)_{m_2} \cdots}{\left(e_1^{(1)}\right)_{m_1} \left(e_2^{(1)}\right)_{m_1} \cdots \left(e_{E^{(1)}}^{(1)}\right)_{m_1} \left(e_1^{(2)}\right)_{m_2} \left(e_2^{(2)}\right)_{m_2} \cdots \left(e_{E^{(2)}}^{(2)}\right)_{m_2} \cdots} \times$$

$$\times \frac{\cdots \left(d_1^{(n)}\right)_{m_n} \left(d_2^{(n)}\right)_{m_n} \cdots \left(d_{D^{(n)}}^{(n)}\right)_{m_n} (x_1)^{m_1}(x_2)^{m_2} \cdots (x_n)^{m_n}}{\cdots \left(e_1^{(n)}\right)_{m_n} \left(e_2^{(n)}\right)_{m_n} \cdots \left(e_{E^{(n)}}^{(n)}\right)_{m_n} m_1!\ m_2!\ \cdots\ m_n!} \sum_{p=0}^{\infty} \frac{(\lambda)_{p+m_1+m_2+\cdots+m_n}}{(\lambda)_{m_1+m_2+\cdots+m_n}} \frac{t^p}{p!}.$$

Using the equation (1.11), we get

$$\Upsilon = \sum_{p=0}^{\infty} \binom{\lambda + p - 1}{p} \times$$

$$\times \sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} \frac{(\lambda + p)_{m_1+m_1+\cdots+m_n} (a_1)_{m_1+m_2+\cdots+m_n} (a_2)_{m_1+m_2+\cdots+m_n} \cdots (a_A)_{m_1+m_2+\cdots+m_n}}{(b_1)_{m_1+m_2+\cdots+m_n} (b_2)_{m_1+m_2+\ldots+m_n} \cdots (b_B)_{m_1+m_2+\cdots+m_n}} \times$$

$$\times \frac{\left(d_1^{(1)}\right)_{m_1} \left(d_2^{(1)}\right)_{m_1} \cdots \left(d_{D^{(1)}}^{(1)}\right)_{m_1} \left(d_1^{(2)}\right)_{m_2} \left(d_2^{(2)}\right)_{m_2} \cdots \left(d_{D^{(2)}}^{(2)}\right)_{m_2} \cdots}{\left(e_1^{(1)}\right)_{m_1} \left(e_2^{(1)}\right)_{m_1} \cdots \left(e_{E^{(1)}}^{(1)}\right)_{m_1} \left(e_1^{(2)}\right)_{m_2} \left(e_2^{(2)}\right)_{m_2} \cdots \left(e_{E^{(2)}}^{(2)}\right)_{m_2} \cdots} \times$$

$$\times \frac{\cdots \left(d_1^{(n)}\right)_{m_n} \left(d_2^{(n)}\right)_{m_n} \cdots \left(d_{D^{(n)}}^{(n)}\right)_{m_n} (x_1)^{m_1}(x_2)^{m_2} \cdots (x_n)^{m_n}}{\cdots \left(e_1^{(n)}\right)_{m_n} \left(e_2^{(n)}\right)_{m_n} \cdots \left(e_{E^{(n)}}^{(n)}\right)_{m_n} m_1!\ m_2!\ \cdots\ m_n!} t^p. \qquad (2.12)$$

Hence, representing the equation (2.12), in hypergeometric form we get the relation (2.7).
Similarly, we get the remaining generating relations (2.1)-(2.4) and (2.8) in the same way as the proof of generating relations (2.5), (2.6) and (2.7).

3. Special Cases

In seventh generating relation (2.7), put $A = B = 0,\ D^{(1)} = D^{(2)} = D^{(3)} = \cdots = D^{(n)} = 1,\ E^{(1)} = E^{(2)} = E^{(3)} = \cdots = E^{(n)} = 1$ and $d_1^{(1)} \equiv d_1,\ d_1^{(2)} \equiv d_2,\ d_1^{(3)} \equiv d_3, ..., d_1^{(n)} \equiv d_n,\ e_1^{(1)} \equiv e_1,\ e_1^{(2)} \equiv e_2,\ e_1^{(3)} \equiv e_3, ..., e_1^{(n)} \equiv e_n$, we get a generating relation

$$(1-t)^{-\lambda} F_A^{(n)} \left[\lambda; d_1, d_2, d_3, ..., d_n; e_1,\ e_2,\ e_3, ...,\ e_n;\ \tfrac{x_1}{1-t}, \tfrac{x_2}{1-t}, \tfrac{x_3}{1-t}, \cdots, \tfrac{x_n}{1-t} \right]$$
$$= \sum_{p=0}^{\infty} \binom{\lambda+p-1}{p} F_A^{(n)} \left[\lambda+p; d_1, d_2, d_3, .., d_n; e_1, e_2, e_3, ..., e_n; x_1, x_2, x_3, ..., x_n \right] t^p, \tag{3.1}$$

where $|t| < 1,\ |\frac{x_1}{1-t}| + |\frac{x_2}{1-t}| + \cdots + |\frac{x_n}{1-t}| < 1$ and $F_A^{(n)}$ is Lauricella function of n variables [16, p.60, Eq.(1)].
When $x_3 = x_4 = \cdots = x_n = 0$ in equation (3.1), and applying the definition of Appell's function F_2 of second kind [3] see also [16, p.53, Eq.(5)], we get a known result [5, p.28, Eq.(2.2)]

$$(1-t)^{-\lambda} F_2 \left[\lambda\ ;\ d_1,\ d_2;\ e_1,\ e_2;\ \tfrac{x_1}{1-t}, \tfrac{x_2}{1-t} \right]$$
$$= \sum_{p=0}^{\infty} \binom{\lambda+p-1}{p} F_2 \left[\lambda+p;\ d_1,\ d_2\ ;\ e_1,\ e_2\ ;\ x_1, x_2 \right] t^p, \tag{3.2}$$

where $|t| < 1,\ |\frac{x_1}{1-t}| + |\frac{x_2}{1-t}| < 1$.

In eighth generating relation (2.8), put $A = 0,\ B = 1,\ D^{(1)} = 1,\ D^{(2)} = D^{(3)} = \cdots = D^{(n)} = 2,\ E^{(1)} = E^{(2)} = E^{(3)} = \cdots = E^{(n)} = 0$ and $d_1^{(1)} \equiv d,\ d_1^{(2)} \equiv g_2,\ d_1^{(3)} \equiv g_3, ..., d_1^{(n)} \equiv g_n,\ d_2^{(2)} \equiv h_2,\ d_2^{(3)} \equiv h_3, ..., d_2^{(n)} \equiv h_n,\ b_1 \equiv b,$ we get a generating relation

$$(1-t)^{-\lambda}\ F_B^{(n)} \left[\lambda,\ g_2,\ g_3, ..., g_n;\ d,\ h_2,\ h_3, ..., h_n;\ b\ ;\ \tfrac{x_1}{1-t}, \tfrac{x_2}{1-t}, \tfrac{x_3}{1-t}, \cdots, \tfrac{x_n}{1-t} \right]$$
$$= \sum_{p=0}^{\infty} \binom{\lambda+p-1}{p} F_B^{(n)} \left[\lambda+p, g_2, g_3, ..., g_n; d, h_2, h_3, ..., h_n; b; x_1, \tfrac{x_2}{1-t}, \tfrac{x_3}{1-t}, \cdots, \tfrac{x_n}{1-t} \right] t^p, \tag{3.3}$$

where $|t| < 1$, $\max\{|\frac{x_1}{1-t}|, |\frac{x_2}{1-t}|, \cdots, |\frac{x_n}{1-t}|\} < 1$ and $F_B^{(n)}$ is Lauricella function of n variables [16, p.60, Eq.(2)].
When $x_3 = x_4 = \cdots = x_n = 0$ in equation (3.3), and applying the definition of Appell's function F_3 of third kind [3] see also [16, p.53, Eq.(6)], we get a known result [5, p.29, Eq.(2.3)]

$$(1-t)^{-\lambda}\, F_3\left[\ \lambda,\ g_2,\ d,\ h_2\ ;\ b\ ;\ \tfrac{x_1}{1-t}, \tfrac{x_2}{1-t}\ \right]$$
$$= \sum_{p=0}^{\infty} \binom{\lambda+p-1}{p} F_3\left[\ \lambda+p,\ g_2,\ d,\ h_2\ ;\ b\ ;\ x_1, \tfrac{x_2}{1-t}\ \right] t^p, \tag{3.4}$$

where $|t| < 1$, $\max\{|\frac{x_1}{1-t}|, |\frac{x_2}{1-t}|\} < 1$.
In seventh generating relation (2.7), put $A = 1,\ B = D^{(1)} = D^{(2)} = D^{(3)} = \cdots = D^{(n)} = 0,\ E^{(1)} = E^{(2)} = E^{(3)} = \cdots = E^{(n)} = 1$ and $e_1^{(1)} \equiv e_1,\ e_1^{(2)} \equiv e_2, ...,\ e_1^{(n)} \equiv e_n,\ a_1 \equiv a$, we get a generating relation

$$(1-t)^{-\lambda} F_C^{(n)}\left[\ \lambda, a; e_1, e_2, e_3, ..., e_n;\ \tfrac{x_1}{1-t}, \tfrac{x_2}{1-t}, \tfrac{x_3}{1-t}, \cdots, \tfrac{x_n}{1-t}\ \right]$$
$$= \sum_{p=0}^{\infty} \binom{\lambda+p-1}{p} F_C^{(n)}\left[\ \lambda+p, a; e_1, e_2, e_3, ..., e_n; x_1, x_2, x_3, ..., x_n\ \right] t^p, \tag{3.5}$$

where $|t| < 1$, $\sqrt{|\frac{x_1}{1-t}|} + \sqrt{|\frac{x_2}{1-t}|} + \cdots + \sqrt{|\frac{x_n}{1-t}|} < 1$ and $F_C^{(n)}$ is Lauricella function of n variables [16, p.60, Eq.(3)].
When $x_3 = x_4 = \cdots = x_n = 0$ in equation (3.5), and applying the definition of Appell's function F_4 of fourth kind [3] see also [16, p.53, Eq.(7)], we get a known result [5, p.29, Eq.(2.4)]

$$(1-t)^{-\lambda}\, F_4\left[\ \lambda,\ a\ ;\ e_1,\ e_2;\ \tfrac{x_1}{1-t}, \tfrac{x_2}{1-t}\ \right]$$
$$= \sum_{p=0}^{\infty} \binom{\lambda+p-1}{p} F_4\left[\ \lambda+p,\ a\ ;\ e_1,\ e_2;\ x_1, x_2\ \right] t^p, \tag{3.6}$$

where $|t| < 1$, $\sqrt{|\frac{x_1}{1-t}|} + \sqrt{|\frac{x_2}{1-t}|} < 1$.
In seventh generating relation (2.7), put $A = 0,\ B = 1,\ D^{(1)} = D^{(2)} = D^{(3)} = \cdots = D^{(n)} = 1,\ E^{(1)} = E^{(2)} = E^{(3)} = \cdots = E^{(n)} = 0$ and $b_1 \equiv b,\ d_1^{(1)} \equiv d_1$, $d_1^{(2)} \equiv d_2, ...,\ d_1^{(n)} \equiv d_n$, we get a generating relation

$$(1-t)^{-\lambda} F_D^{(n)}\left[\ \lambda; d_1, d_2, d_3, ..., d_n; b;\ \tfrac{x_1}{1-t}, \tfrac{x_2}{1-t}, \tfrac{x_3}{1-t}, ..., \tfrac{x_n}{1-t}\ \right]$$
$$= \sum_{p=0}^{\infty} \binom{\lambda+p-1}{p} F_D^{(n)}\left[\ \lambda+p; d_1, d_2,\ d_3, ..., d_n; b; x_1\ x_2, x_3, ..., x_n\ \right] t^p, \tag{3.7}$$

where $|t| < 1$, $\max\{|\frac{x_1}{1-t}|, |\frac{x_2}{1-t}|, \cdots, |\frac{x_n}{1-t}|\} < 1$ and $F_D^{(n)}$ is Lauricella function of n variables [16, p.60, Eq.(4)].
When $x_3 = x_4 = \cdots = x_n = 0$ in equation (3.7), and applying the definition of Appell's function F_1 of first kind [3] see also [16, p.53, Eq.(4)], we get a known result [5, p.28, Eq.(2.1)]

$$(1-t)^{-\lambda} F_1 \left[\lambda;\ d_1,\ d_2\ ;\ b\ ;\ \tfrac{x_1}{1-t}, \tfrac{x_2}{1-t} \right]$$
$$= \sum_{p=0}^{\infty} \binom{\lambda+p-1}{p} F_1 \left[\lambda+p;\ d_1,\ d_2\ ;\ b\ ; x_1, x_2 \right] t^p, \tag{3.8}$$

where $|t| < 1$, $\max\{|\frac{x_1}{1-t}|, |\frac{x_2}{1-t}|\} < 1$.

4. Conclusion

We conclude the present investigation by noting that the family of Bessel functions is introduced in standard form. The results derived in this paper are quite significant and general in character. Moreover, we derive some generating relations, involving multiple hypergeometric functions of Kampé de Fériet with some special cases. We remark that the results obtained in current paper are expected to lead some potential applications in several diverse fields of mathematical, physical, statistical and engineering sciences.

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